

# Sobolev–type metrics in the space of curves

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## Abstract

We define a manifold  $M$  where objects  $c \in M$  are curves, which we parameterize as  $c : S^1 \rightarrow \mathbb{R}^n$  ( $n \geq 2$ ,  $S^1$  is the circle). Given a curve  $c$ , we define the tangent space  $T_c M$  of  $M$  at  $c$  including in it all deformations  $h : S^1 \rightarrow \mathbb{R}^n$  of  $c$ .

In this paper we study geometries on the manifold of curves, provided by Sobolev–type metrics  $H^j$ . We study  $H^j$  type metrics for the cases  $j = 1, 2$ ; we prove estimates, and characterize the completion of the space of smooth curves.

As a bonus, we prove that the Fréchet distance of curves (see [MM06b]) coincides with the distance induced by the “Finsler  $L^\infty$  metric” defined in §2.2 in [YM04b].

## 1 Introduction

Suppose that  $c$  is an immersed curve  $c : S^1 \rightarrow \mathbb{R}^n$ , where  $S^1 \subset \mathbb{R}^2$  is the circle; we want to define a geometry on  $M$ , the space of all such immersions  $c$ .

The tangent space  $T_c M$  of  $M$  at  $c$  contains all the *deformations*  $h \in T_c M$  of the curve  $c$ , that are all the vector fields along  $c$ . Then, an infinitesimal deformation of the curve  $c$  in “direction”  $h$  will yield (on first order) the curve  $c(u) + \varepsilon h(u)$ .

For the sake of simplicity, we postpone details of the definitions (in particular on the regularity of  $c$  and  $h$ , and the topology on  $M$ ) to §2.

We would like to define a *Riemannian metric* on the manifold  $M$  of immersed curves: this means that, given two deformations  $h, k \in T_c M$ , we want to define a scalar product  $\langle h, k \rangle_c$ , possibly dependent on  $c$ . The Riemannian metric would then entail a *distance*  $d(c_0, c_1)$  between the curves in  $M$ , defined as the infimum of the length  $\text{Len}(\gamma)$  of all smooth paths  $\gamma : [0, 1] \rightarrow M$  connecting  $c_0$  to  $c_1$ . We call *minimal geodesic* a path providing the minimum of  $\text{Len}(\gamma)$  in the class of  $\gamma$  with fixed endpoints. <sup>(1)</sup>

At the same time, we would like to consider the curves as “geometric objects”; to this end, we will define the space of *geometrical curves*  $B \stackrel{\text{def}}{=} M / \text{Diff}(S^1)$ , that is the space of immersed curves up to reparametrization. To this end, we will ask that the metric (and all the energies) defined on  $M$  be independent of the parameterization of the curves.

$B$  and  $M$  are the *Shape Spaces* that are studied in this paper.

A number of methods have been proposed in Shape Analysis to define distances between shapes, averages of shapes and optimal morphings between shapes. At the

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<sup>(1)</sup>Note that this is an oversimplification of what we will actually do: compare definitions 8 and 10

same time, there has been much previous work in Shape Optimization, for example Image Segmentation via Active Contours, 3D Stereo Reconstruction via Deformable Surfaces; in these later methods, many authors have defined Energy Functionals  $E(c)$  on curves (or on surfaces), whose minima represent the desired Segmentation/Reconstruction; and then utilized the Calculus of Variations to derive curve evolutions to search minima of  $E(c)$ , often referring to these evolutions as Gradient Flows. The reference to these flows as *gradient flows* implies a certain Riemannian metric on the space of curves; but this fact has been largely overlooked. We call this metric  $H^0$ , and define it by

$$\langle h, k \rangle_{H^0} \stackrel{\text{def}}{=} \frac{1}{L} \int_{S^1} \langle h(s), k(s) \rangle \, ds$$

where  $h, k \in T_c M$ ,  $L$  is the length of  $c$ ,  $ds \stackrel{\text{def}}{=} |\dot{c}(\theta)| \, d\theta$  is integration by arc-parameter, and  $\langle h(s), k(s) \rangle$  is the usual Euclidian scalar product in  $\mathbb{R}^n$  (that sometimes we will also write as  $h(s) \cdot k(s)$ ).

For example, the well known Geometric Heat Flow is often referred as the *gradient flow for length*: we show how it is indeed the gradient flow w.r.t. the  $H^0$  metric.

**Example 1** Let  $c$  be an immersed curve, and  $h$  be a deformation of  $c$ . Let the differential operator  $D_s \stackrel{\text{def}}{=} \frac{1}{|c|} \partial_\theta$  be “the derivative with respect to arclength”. Let

$$\text{len}(c) \stackrel{\text{def}}{=} \int_{S^1} |\dot{c}(\theta)| \, d\theta \quad (1)$$

be the length of the curve; we recall that

$$\frac{\partial \text{len}(c)}{\partial h} = \int_{S^1} \langle D_s h \cdot T \rangle \, ds = - \int_{S^1} \langle h \cdot D_s^2 c \rangle \, ds \quad (2)$$

where  $T = D_s c$  is the tangent to the curve,  $D_s^2 c$  is the curvature, intended as a vector.

Let  $C = C(\theta, t)$  be an evolving family of curves trying to minimize  $L$ : we recall moreover that the resulting geometric heat flow (also known as motion by mean curvature)

$$\frac{\partial C}{\partial t} = D_s^2 C$$

is well defined only for positive times.

By comparing the above flow to the definition of  $H^0$ , we realize that this flow is the gradient descent (up to a conformal term  $1/\text{len}(c)$ ):

$$\frac{\partial C}{\partial t} = - \frac{1}{\text{len}(c)} \nabla_{H^0} \text{len}(c)$$

If one wishes to have a consistent view of the geometry of the space of curves in both Shape Optimization and Shape Analysis, then one should use the  $H^0$  metric when computing distances, averages and morphs between shapes.

Surprisingly,  $H^0$  does not yield a well define metric structure, since the associated distance is identically zero<sup>(2)</sup>.

Moreover, some simple Shape Optimization tasks are ill-defined when using the  $H^0$  metric:

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<sup>(2)</sup>This striking fact was first described in [Mum]; it is generalized to spaces of submanifolds in [MM05]

**Remark 2** Let  $g : \mathbb{R}^n \rightarrow \mathbb{R}^k$ , let

$$\text{avg}(g(c)) \stackrel{\text{def}}{=} \frac{1}{L} \int_{S^1} g(c(s)) \, ds = \frac{1}{L} \int_{S^1} g(c(\theta)) |\dot{c}(\theta)| \, d\theta ;$$

(here again  $L \stackrel{\text{def}}{=} \text{len}(c)$ ); then

$$\begin{aligned} \frac{\partial \text{avg}(g(c))}{\partial h} &= \frac{1}{L} \int_{S^1} \nabla g(c) h + g(c) \langle D_s h \cdot T \rangle \, ds - \\ &\quad - \frac{1}{L^2} \int_{S^1} g(c) \, ds \int_{S^1} \langle D_s h \cdot T \rangle \, ds = \\ &= \frac{1}{L} \int_{S^1} \nabla g(c) h + (g(c) - \text{avg}(g(c))) \langle D_s h \cdot T \rangle \, ds \end{aligned} \quad (3)$$

If the curve is in the plane, that is  $n = 2$ , then we define the normal vector  $N \perp T$  by rotating  $T$  counterclockwise, and define scalar curvature  $\kappa$  so that  $D_s^2 c = \kappa N$ ; then, integrating by parts, the above becomes

$$\begin{aligned} \frac{\partial \text{avg}(g(c))}{\partial h} &= \frac{1}{L} \int_{S^1} \frac{\partial g}{\partial x}(c) h - \left( \frac{\partial g}{\partial x}(c(s)) T \right) \langle h \cdot T \rangle - \\ &\quad - (g(c) - \text{avg}(g(c))) \langle h \cdot D_s^2 c \rangle \, ds = \\ &= \frac{1}{L} \int_{S^1} \left( \frac{\partial g}{\partial x}(c) N - \kappa (g(c) - \text{avg}(g(c))) \right) \langle h \cdot N \rangle \, ds \end{aligned}$$

Suppose now that we have a Shape Optimization functional  $E$  including a term of the form  $\text{avg}(g(c))$ ; let  $C = C(\theta, t)$  be an evolving family of curves trying to minimize  $E$ ; this flow would contain a term of the form

$$\frac{\partial C}{\partial t} = \dots (g(c(s)) - \bar{g}) \kappa N \dots$$

unfortunately the above flow is ill defined: it is a negative-time heat flow on roughly half of the curve. We present two simple examples.

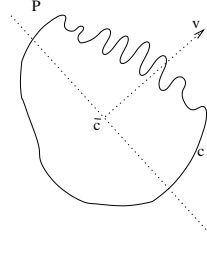
- If for example  $g(x) = x$ , then  $\text{avg}(g(c)) = \text{avg}(c)$  is the **center of mass of the curve**. Let us fix a target point  $v \in \mathbb{R}^2$ . Let  $E(c) \stackrel{\text{def}}{=} \frac{1}{2} |\text{avg}(c) - v|^2$  be a functional that penalizes the distance from the center of mass to  $v$ ; by direct computation

$$\begin{aligned} \frac{\partial E}{\partial h} &= (\text{avg}(c) - v) \cdot \frac{\partial \text{avg}(c)}{\partial h} = \\ &= \frac{1}{L} \int_{S^1} \langle (\bar{c} - v) \cdot (N - \kappa(c - \bar{c})) \rangle \langle h \cdot N \rangle \, ds \end{aligned}$$

(where  $\bar{c} = \text{avg}(c)$  for simplicity); so we conclude that the  $H^0$  gradient descent flow is

$$\frac{\partial C}{\partial t} = -\nabla_{H^0} E(c) = \langle (v - \bar{c}) \cdot N \rangle N - \kappa N \langle (c - \bar{c}) \cdot (v - \bar{c}) \rangle \quad (4)$$

(up to a part tangent to the curve).



Let  $P \stackrel{\text{def}}{=} \{w : \langle (w - \bar{c}) \cdot (v - \bar{c}) \rangle \geq 0\}$  be the halfplane that is the region of the plane that is “on the  $v$  side” w.r.t.  $\bar{c}$ . This gradient descent flow (4) does move the center of mass towards the point  $v$ : indeed there is a first term  $\langle (v - \bar{c}) \cdot N \rangle N$  that moves the whole curve towards  $v$ ; and a second term that tries to decrease the curve length out of  $P$  and increase the curve length in  $P$ : and this is ill posed.

- Similarly if  $g(x) = |x - \bar{c}|^2$ , then  $E(c) \stackrel{\text{def}}{=} \text{avg}(|c - \bar{c}|^2)$  is the **standard deviation of the curve**. The derivative is

$$\frac{2}{L} \int_{S^1} \left( \langle (c - \bar{c}) \cdot N \rangle - \kappa(g(c) - \text{avg}(g(c))) \right) \langle h \cdot N \rangle ds$$

The flow to minimize this should be

$$\frac{\partial C}{\partial t} = -\langle (c - \bar{c}) \cdot N \rangle N + \kappa N(g(c) - \text{avg}(g(c)))$$

and this is ill posed where the curve is inside of the circle of center  $\text{avg}(c)$  and radius  $\sqrt{\text{avg}(g(c))}$

The above phenomenon is also visible in many applications, where the Active Contour curve would “fractalize” in an attempt to minimize the task energy. For this reason, a regularization term is often added to the energy: this remedy, though, does change the energy, and ends up solving a different problem.

### §1.i Sobolev–type Riemannian Metrics

To overcome this limitation, in [SYM05a] and [SYM06] we proposed a family of Sobolev–type Riemannian Metrics

**Definition 3** Let  $c \in M$ ,  $L$  be the length of  $c$ , and  $h, k \in T_c M$ . Let  $\lambda > 0$ . We assume  $h$  and  $k$  are parameterized by the arclength parameter of  $c$ . We define, for  $j \geq 1$  integer,

$$i). \langle h, k \rangle_{H^j} \stackrel{\text{def}}{=} \langle h, k \rangle_{H^0} + \lambda L^{2j} \langle D_s^j h, D_s^j k \rangle_{H^0}$$

$$ii). \langle h, k \rangle_{\tilde{H}^j} \stackrel{\text{def}}{=} \text{avg}(h) \cdot \text{avg}(k) + \lambda L^{2j} \langle D_s^j h, D_s^j k \rangle_{H^0}$$

where again  $\text{avg}(h) \stackrel{\text{def}}{=} \frac{1}{L} \int_{S^1} h(s) ds$  and  $D_s^j$  is the  $j$ -th derivative with respect to arclength. <sup>(3)</sup>

It is easy to verify that the above definitions are inner products. Note that we have introduced length dependent scale factors so that the above inner products (and corresponding norms) are independent of curve rescaling.

Changing the metric will change the gradient and thus the gradient descent flow; this change will alter the topology in the space of curves, but the change of topology does not affect the energy to be minimized, or its global minima; whereas it may regularize the flows, and avoid that the flows be trapped in local minima; many examples and applications can be found in the survey [SYM05b].

<sup>(3)</sup>Note that  $\langle h, k \rangle_{H^0} = \text{avg}(h \cdot k)$  so the difference in the two metrics is in using  $\text{avg}(h \cdot k)$  instead of  $\text{avg}(h) \cdot \text{avg}(k)$

## §1.ii Previous work

In [YM04a, YM04b, YM05] we addressed the problem of defining a metric in the space  $M$  of parameterized immersed curves  $c : S^1 \rightarrow \mathbb{R}^n$  (with special attention to the case  $n = 2$ ); we discuss the general setting of Finsler and Riemannian metrics, and related problematics; we propose a set of goal properties. we discuss some models available in the literature. Eventually we proposed and study conformal metrics such as

$$\langle h, k \rangle_{H_\phi^0} \stackrel{\text{def}}{=} \text{len}(c) \int \langle h(s), k(s) \rangle ds \quad (5)$$

we prove results regarding this metric, and in particular, that the associate distance is non degenerate. We also proved that, supposing that only unit length curves with an upper bound on curvature are allowed, then there exist minimal geodesics.

The same approach was proposed independently by J. Shah in [Sha05], who moreover proved that in the simplest case given by (5), the only minimal geodesics are given by a curve evolution whose velocity is proportional to the curve normal vector field.

The problem has been addressed by Michor and Mumford in [MM06b], who propose the metric

$$\langle h, k \rangle_{H_A^0} \stackrel{\text{def}}{=} \int (1 + A\kappa^2(s)) \langle h(s), k(s) \rangle ds \quad (6)$$

where  $\kappa$  is the curvature of  $c$ , and  $A > 0$  is a fixed constant; they prove many results regarding this metric; in particular, that the associate distance is non degenerate, and that completion of smooth curves is contained between the space  $\text{Lip}$  of rectifiable curves, and the space  $\text{BV}^2$  of rectifiable curves whose curvature is a bounded measure.

More recently in [SYM05a] and [SYM06] we studied the family of Sobolev-type metrics defined in 3.

In [SYM05a] we have experimentally shown that Sobolev flows are smooth in the space of curves, are not as dependent on local image information as  $H^0$  flows, are global motions which deform locally after moving globally, and do not require derivatives of the curve to be defined for region-based and edge-based energies. In general, Sobolev gradients can be expressed in terms of the traditional  $H^0$  gradient, that is, we have the formulas

$$\nabla_{H^n} E = \nabla_{H^0} E * K_{\lambda,n} \quad (7)$$

$$\nabla_{\tilde{H}^n} E = \nabla_{H^0} E * \tilde{K}_{\lambda,n} \quad (8)$$

for suitable convolutional kernels  $\tilde{K}_{\lambda,n}, K_{\lambda,n}$ . We have moreover shown mathematically that the Sobolev-type gradients regularize the flows of well known energies, by reducing the degree of the P.D.E.

**Example 4** For example, in the case of the elastic energy  $E(c) = \int_c \kappa^2 ds = \int_c |D_s^2 c|^2 ds$ , the  $H^0$  gradient is  $\nabla_{H^0} E = LD_s(2D_s^{(3)}c + 3|D_s^2 c|^2 D_s c)$  that includes fourth order derivatives; whereas the  $\tilde{H}^1$ -gradient is

$$Lc_{ss} + L(|D_{ss}c|^2 D_s c) * \tilde{K}_{\lambda,1} \quad (9)$$

that is an integro-differential second order P.D.E.

In [SYM06] we have shown numerical experiments on real-life cases, and shown that the regularizing properties may be explained in the Fourier domain: indeed, if we

calculate Sobolev gradients  $\nabla_{H^n} E$  of an arbitrary energy  $E$  in the frequency domain, then

$$\widehat{\nabla_{H^n} E}(l) = \frac{\widehat{\nabla_{H^0} E}(l)}{1 + \lambda(2\pi l)^{2n}} \quad \text{for } l \in \mathbb{Z} \quad (10)$$

and

$$\widehat{\nabla_{H^n} E}(0) = \widehat{\nabla_{H^0} E}(0), \quad \widehat{\nabla_{H^n} E}(l) = \frac{\widehat{\nabla_{H^0} E}(l)}{\lambda(2\pi l)^{2n}} \quad \text{for } l \in \mathbb{Z} \setminus \{0\}, \quad (11)$$

(see eqn. 13 for the precise definition of Fourier coefficients). It is clear from the previous expressions that high frequency components of  $\nabla_{H^0} E(c)$  are increasingly less pronounced in the various forms of the  $H^n$  gradients.

A family of metrics similar to the first example above (but for the length dependent scale factors) is currently studied in [MM06a]: the Sobolev-type weak Riemannian metric on  $\text{Imm}(S^1, \mathbb{R}^2)$

$$\begin{aligned} \langle h, k \rangle_{G_c^n} &= \int_{S^1} \sum_{i=0}^n \langle D_s^i h, D_s^i k \rangle ds = \int_{S^1} \langle A_n(h), k \rangle ds \quad \text{where} \\ A_n(h) &= A_{n,c}(h) = \sum_{i=0}^n (-1)^i D_s^{2i}(h) \end{aligned}$$

in that paper the geodesic equation, horizontality, conserved momenta, lower and upper bounds on the induced distance, and scalar curvatures are computed. Note that this norm is equivalent to the norms in 3: see remark 19.

Many other approaches to Shape Analysis are present in the literature; for example, much earlier than the above, Younes in [You98] had proposed a computable definition of distance of curves, modeled on elastic curves.

A different approach to the study of shapes is obtained when the shape is defined to be a *curve up to reparametrization, rotation, translation, scaling*.

For example Mio, Srivastava et al in [KSMJ03, MS04] use a different choice of curve representation: they represent a planar curve  $c$  by a pair of angle-velocity functions  $\dot{c}(u) = \exp(\phi(u) + i\theta(u))$  (identifying  $\mathbb{R}^2 = \mathbb{C}$ ), and then defining a metric on  $(\phi, \theta)$ . They propose models of spaces of curves where the metrics involve higher order derivatives in [KSMJ03]. See the proof of thm. 24 for an example comparison of the two approaches.

## 2 Spaces of curves

As anticipated in the introduction, we want to define a geometry on  $M$ , the space of all immersions  $c : S^1 \rightarrow \mathbb{R}^n$ .

We will sometimes distinguish exactly what  $M$  is, choosing between the space  $\text{Imm}(S^1, \mathbb{R}^n)$  of immersions, the space  $\text{Imm}_f(S^1, \mathbb{R}^n)$  of *free immersions*, and  $\text{Emb}(S^1, \mathbb{R}^n)$  of *embeddings* (see §2.4, §2.5 in [MM06b]).

We will equip  $M$  with a topology  $\tau$  stronger than the  $C^1$  topology: then any such choice  $M$  is an open subset of the vector space  $C^1(S^1, \mathbb{R}^n)$  (that is a Banach space), so it is a manifold.

The tangent space  $T_c M$  of  $M$  at  $c$  contains vector fields  $h : S^1 \rightarrow \mathbb{R}^n$  along  $c$ .

Note that we represent both curves  $c \in M$  and deformations  $h \in T_c M$  as functions  $S^1 \rightarrow \mathbb{R}^n$ ; this is a special structure that is not usually present in abstract manifolds: so we can easily define “charts” for  $M$ :

**Remark 5 (Charts in  $M$ )** *Given a curve  $c$ , there is a neighbourhood  $U_c$  of  $0 \in T_c M$  such that for  $h \in U_c$ , the curve  $c + h$  is still immersed; then this map  $h \mapsto c + h$  is the simplest natural candidate to be a chart of  $\Phi_c : U_c \rightarrow M$ ; indeed, if we pick another curve  $\tilde{c} \in M$  and the corresponding  $U_{\tilde{c}}$  such that  $U_{\tilde{c}} \cap U_c \neq \emptyset$ , then the equality  $\Phi_c(h) = c + h = \tilde{c} + \tilde{h} = \Phi_{\tilde{c}}(\tilde{h})$  can be solved for  $h$  to obtain  $h = (\tilde{c} - c) + \tilde{h}$ .*

The above is trivial but is worth remarking for two reasons: it stresses that the topology  $\tau$  must be strong enough to maintain immersions; and is a basis block to what we will do in the space  $B_{i,f}$  defined below.

We look mainly for metrics in the space  $M$  that are independent on the parameterization of the curves  $c$ : to this end, we define these spaces of *geometrical curves*

$$B_i = B_i(S^1, \mathbb{R}^2) = \text{Imm}(S^1, \mathbb{R}^2)/\text{Diff}(S^1)$$

and

$$B_{i,f} = B_{i,f}(S^1, \mathbb{R}^2) = \text{Imm}_f(S^1, \mathbb{R}^2)/\text{Diff}(S^1)$$

that are the quotients of the spaces  $\text{Imm}_f$  and  $\text{Imm}_f(S^1, \mathbb{R}^2)$  by  $\text{Diff}(S^1)$ ; alternatively we may quotient by  $\text{Diff}^+(S^1)$  (the space of orientation preserving automorphisms of  $S^1$ ), and obtain spaces of *geometrical oriented curves*.

**Remark 6 (on model spaces and properties)** *We have two possible choices in mind for the topology  $\tau$  to put on  $M$ : the Fréchet space of  $C^\infty$  functions; or a Hilbert space such as standard Sobolev space  $H^j(S^1 \rightarrow \mathbb{R}^n)$ .*

*Suppose we define on  $M$  a Riemannian metric: we would like  $B_i$  to have a nice geometrical structure; we would like our Riemannian Geometry to satisfy some useful properties.*

*Unfortunately, this currently seems an antinomy.*

*If  $M$  is modeled on a Hilbert space  $H^j$ , then most of the usual calculus carries on; for example, the exponential map would be locally a diffeomorphism; but the quotient space  $M/\text{Diff}(S^1)$  is not a smooth bundle, (since the tangent to the orbit contains  $\dot{c}$  and this is in  $H^{j-1}$  in general!).*

*If  $M$  is modeled on the Fréchet space of  $C^\infty$  functions, then the quotient space  $M/\text{Diff}(S^1)$  is a smooth bundle; but some of the usual calculus fails: the Cauchy theorem does not hold in general; and the exponential is not locally surjective.*

Suppose in the following that  $\tau$  is the Fréchet space of  $C^\infty$  functions, for simplicity; then  $B_{i,f}$  is a manifold, the base of a principal fiber bundle while  $B_i$  is not (see in §2.4.3 in [MM06b]).

To define charts on this manifold, we imitate what was done for  $M$ :

**Proposition 7 (Charts in  $B_{i,f}$ )** *Let  $\Pi$  be the projection from  $\text{Imm}_f(S^1, \mathbb{R}^2)$  to the quotient  $B_{i,f}$ .*

*Let  $[c] \in B_{i,f}$ : we pick a curve  $c$  such that  $\Pi(c) = [c]$ . We represent the tangent space  $T_{[c]}B_{i,f}$  as the space of all  $k : S^1 \rightarrow \mathbb{R}^n$  such that  $k(s)$  is orthogonal to  $\dot{c}(s)$ .*

*Again we can define a simple natural chart  $\Phi_{[c]}$  by projecting the chart  $\Phi_c$  (defined in 5): the chart is*

$$\Phi_{[c]}(k) \stackrel{\text{def}}{=} \Pi(c(\cdot) + k(\cdot))$$

that is, it moves  $c(u)$  in direction  $k(u)$ ; and it is easily seen that the chart does not depend on the choice of  $c$  such that  $\Pi(c) = [c]$ . We can solve  $\Phi_{[c]}(k) = \Phi_{[\tilde{c}]}(\tilde{k})$  (this is not so easy to prove: see [MM06b], or 4.4.7 and 4.6.6 in [Ham82]).

Define a Finsler metric  $F$  on  $M$ ; this is a lower semi continuous function such that  $F(c, \cdot)$  is a norm on  $T_c M$ .

If  $\gamma : [0, 1] \rightarrow M$  is a path connecting two curves  $c_0, c_1$ , then we may define a homotopy  $C : S^1 \times [0, 1] \rightarrow \mathbb{R}^n$  associated to  $\gamma$  by  $C(\theta, v) = \gamma(v)(\theta)$ , and viceversa.

**Definition 8 (standard distance)** Given a metric  $F$  in  $M$ , we could consequently define the standard distance of two curves  $c_0, c_1$  as the infimum of the length

$$\int_0^1 F(\gamma(t), \dot{\gamma}(t)) dt$$

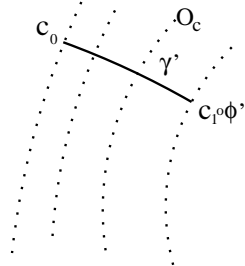
in the class of all  $\gamma$  connecting  $c_0, c_1$ .

This is not, though, the most interesting distance for applications: we are indeed interested in studying metrics and distances in the quotient space  $B \stackrel{\text{def}}{=} M/\text{Diff}(S^1)$ .

We suppose that

**Definition 9** the metric  $F(c, h)$  is “**curve-wise parameterization invariant**”, that is, it does not depend on the parameterization of the curves  $c$

then  $F$  may be projected to  $B \stackrel{\text{def}}{=} M/\text{Diff}(S^1)$ ; we will say that  $F$  is a **geometrical metric**.



Consider two geometrical curves  $[c_0], [c_1] \in B$ , and a path  $\gamma : [0, 1] \rightarrow B$  connecting  $[c_0], [c_1]$ : then we may lift it to a homotopy  $C : S^1 \times [0, 1] \rightarrow \mathbb{R}^n$ ; in this case, the homotopy will connect a  $c_0 \circ \phi_0$  to  $c_1 \circ \phi_1$ , with  $\phi_0, \phi_1 \in \text{Diff}(S^1)$ . Since  $F$  does not depend on the parameterization, we can factor out  $\phi_0$  from the definition of the projected length.

To summarize, we define the

**Definition 10 (geometric distance)** Given  $c_0, c_1$ , we define the class  $\mathcal{A}$  of homotopies  $C$  connecting the curve  $c_0$  to a reparameterization <sup>(4)</sup>  $c_1 \circ \phi$  of the curve  $c_1$ , that is,  $C(u, 0) = c_0(u)$  and  $C(u, 1) = c_1(\phi(u))$ . We define the geometric distance  $d_F$  of  $[c_0], [c_1]$  in  $B \stackrel{\text{def}}{=} M/\text{Diff}(S^1)$  as the infimum of the length

$$\text{Len}_F(C) \stackrel{\text{def}}{=} \int_0^1 F(C(\cdot, v), \partial_v C(\cdot, v)) dv$$

in the class of all such  $C \in \mathcal{A}$ . <sup>(5)</sup>

Any homotopy that achieves the minimum of  $\text{Len}_F(C)$  is called a geodesic.

We call such distances  $d_F(c_0, c_1)$ , dropping the square brackets for simplicity. <sup>(6)</sup>

We provide an interesting example of the above ideas.

<sup>(4)</sup>If we use  $\text{Diff}^+(S^1)$  to define  $B$  then  $\phi$  must be orientation preserving as well.

<sup>(5)</sup>Note the difference between  $\text{Len}(C)$  and  $\text{len}(c)$ , that was defined in eqn. (1).

<sup>(6)</sup>We are abusing notation: these  $d_F$  are not, properly speaking, distances in the space  $M$ , since the distance between  $c$  and a reparameterization  $c \circ \phi$  is zero.

## §2.i $L^\infty$ -type Finsler metric and Fréchet distance

We digress from the main theme of the paper to prove a result that will be used in the following.

For any fixed immersed curve  $c$  and  $\theta \in S^1$ , we define for convenience  $\pi_N : \mathbb{R}^n \rightarrow \mathbb{R}^n$  to be the projection on the space  $N(\theta)$  orthogonal to the tangent vector  $D_s c(\theta)$ ,

$$\pi_{N(\theta)} w = w - \langle w, D_s c(\theta) \rangle D_s c(\theta) \quad \forall w \in \mathbb{R}^n. \quad (12)$$

Consider two immersed curves  $c_0$  and  $c_1$ ; the Fréchet distance  $d_f$  (as found in [MM06b]) is defined by

**Definition 11 (Fréchet distance)**

$$d_f(c_0, c_1) \stackrel{\text{def}}{=} \inf_{\phi} \sup_u |c_1(\phi(u)) - c_0(u)|$$

where  $u \in S^1$  and  $\phi$  is chosen in the class of diffeomorphisms of  $S^1$ .

This is a well defined distance in the space  $B_i$  (that is not, though, complete w.r.t. this distance: its completion is the space of Fréchet curves).

Another similar distance was defined in §2.2 in [YM04b] by a different approach, using a Finsler metric:

**Definition 12 (Finsler  $L^\infty$  metric)** *If we wish to define a norm  $F(c, \cdot)$  on  $T_c M$  that is modeled on the norm of the Banach space  $L^\infty(S^1 \rightarrow \mathbb{R}^n)$ , we define*

$$F^\infty(c, h) \stackrel{\text{def}}{=} \|\pi_N h\|_{L^\infty} = \sup_{\theta} |\pi_{N(\theta)} h(\theta)|$$

We define the distance  $d_\infty(c_0, c_1)$  as in 10.

Section §2.2.1 in [YM04b] discusses the relationship between the distance  $d_\infty$  and the Hausdorff distance of compact sets; we instead discuss the relationship between  $d_f$  and  $d_\infty$ : indeed we prove that  $d_f = d_\infty$ .

**Theorem 13**  $d_f = d_\infty$ .

*Proof.* Fix  $c_0$  and  $c_1$ , and define  $\mathcal{A}$  as in 10.

We recall that  $d_\infty$  is also equal to the infimum of

$$d_\infty(c_0, c_1) = \inf_{C \in \mathcal{A}} \int_0^1 \sup_{\theta} \left| \frac{\partial C}{\partial v}(\theta, v) \right| dv$$

as well (the proof follows immediatly from prop. 3.10 in [YM04b])

Consider a homotopy  $C = C(u, v) \in \mathcal{A}$  connecting the curve  $c_0$  to a reparameterization  $c_1 \circ \phi$  of the curve  $c_1$ :

$$\begin{aligned} \sup_u |c_1(\phi(u)) - c_0(u)| &= \sup_u |C(u, 1) - C(u, 0)| = \\ &= \sup_u \left| \int_0^1 \frac{\partial C}{\partial v}(u, v) dv \right| \leq \int \sup_u \left| \frac{\partial C}{\partial v}(u, v) \right| dv \end{aligned}$$

so that  $d_f \leq d_\infty$ .

On the other side, let

$$C^\phi(\theta, v) \stackrel{\text{def}}{=} (1-v)c_0(\theta) + vc_1(\phi(\theta))$$

be the linear interpolation: then

$$\frac{\partial C^\phi}{\partial v}(u, v) = c_1(\phi(u)) - c_0(u)$$

(that does not depend on  $v$ ) so that

$$\sup_u \left| \int_0^1 \frac{\partial C^\phi}{\partial v}(u, v) dv \right| = \int \sup_u \left| \frac{\partial C^\phi}{\partial v}(u, v) \right| dv$$

and then, for that particular homotopy  $C^\phi$ ,

$$\text{Len}_\infty(C^\phi) = \sup_u |c_1(\phi(u)) - c_0(u)|$$

we compute the infimum of all possible choices of  $\phi$  and get that

$$d_\infty(c_0, c_1) = \inf_C \text{Len}_\infty(C) \leq \inf_\phi \text{Len}_\infty(C^\phi) = \inf_\phi \sup_u |c_1(\phi(u)) - c_0(u)| = d_f(c_0, c_1)$$

□

The theorem holds as well if use orientation preserving diffeomorphism  $\text{Diff}^+(S^1)$  both in the definition of the Fréchet distance and in the definition of  $L^\infty$ .

### 3 Sobolev-type $H^j$ metrics

We start by generalizing the definition in 3. Fix  $\lambda > 0$ . Suppose that  $h \in L^2$ , then we can express it in Fourier series:

$$h(s) = \sum_{l \in \mathbb{Z}} \hat{h}(l) \exp\left(\frac{2\pi i}{L} ls\right) \quad (13)$$

where  $\hat{h} \in \ell^2(\mathbb{Z} \rightarrow \mathbb{C})$ ; and similarly for  $k$ .

For any  $\alpha > 0$ , given the Fourier coefficients  $\hat{h}, \hat{k} : \mathbb{Z} \rightarrow \mathbb{C}$  of  $h, k$ , we define the fractional Sobolev inner product

$$\langle h, k \rangle_{H_0^\alpha} \stackrel{\text{def}}{=} \sum_{l \in \mathbb{Z}} (2\pi l)^{2\alpha} \hat{h}(l) \cdot \overline{\hat{k}(l)} \quad (14)$$

that is independent of curve scaling; then we can define

$$\begin{aligned} \langle h, k \rangle_{H^\alpha} &\stackrel{\text{def}}{=} \text{avg}(h \cdot k) + \lambda \langle h, k \rangle_{H_0^\alpha} \\ \langle h, k \rangle_{\tilde{H}^\alpha} &\stackrel{\text{def}}{=} \text{avg}(h) \cdot \text{avg}(k) + \lambda \langle h, k \rangle_{H_0^\alpha} \end{aligned} \quad (15)$$

When  $\alpha = j$  integer, these definition coincide with the one in 3. So, for any  $\alpha > 0$ , we represent the Sobolev-type metrics by

$$\langle h, k \rangle_{H^\alpha} = \sum_{l \in \mathbb{Z}} (1 + \lambda(2\pi l)^{2\alpha}) \hat{h}(l) \cdot \overline{\hat{k}(l)} \quad (16)$$

$$\langle h, k \rangle_{\tilde{H}^\alpha} = \hat{h}(0) \cdot \overline{\hat{k}(0)} + \sum_{l \in \mathbb{Z}} \lambda(2\pi l)^{2\alpha} \hat{h}(l) \cdot \overline{\hat{k}(l)}. \quad (17)$$

**Remark 14** Unfortunately for  $j$  that is not an integer, the inner products (therefore, norms) are not local, that is, they cannot be written as integrals of derivatives of the curves. An interesting representation is by kernel convolution: given  $r \in \mathbb{R}^+$ , we can represent them, for  $j$  integer  $j > r + 1/4$ , as

$$\langle h, k \rangle_{\tilde{H}^r} = \int_c \int_c D^j h(s) K(s - \tilde{s}) D^j k(\tilde{s}) \, ds \, d\tilde{s}$$

that is,  $\langle h, k \rangle_{\tilde{H}^r} = \langle D^j h, K * D^j k \rangle_{H^0}$ , for a specific kernel  $K$ , ( $*$  denotes convolution in  $S^1$  w.r.t. arc parameter).

**Remark 15** The norm  $\|h\|_{\tilde{H}^j}$  has an interesting interpretation in connection with applications in Computer Vision.

Consider a deformation  $h \in T_c M$  and write it as  $h = \text{avg}(h) + \tilde{h}$ : this decomposes

$$T_c M = \mathbb{R}^n \oplus D_c M \quad (18)$$

with

$$D_c M \stackrel{\text{def}}{=} \left\{ h : S^1 \rightarrow \mathbb{R}^n \mid \text{avg}(h) = 0 \right\}$$

If we assign to  $\mathbb{R}^n$  its usual euclidean norm, and to  $D_c M$  the scale-invariant  $H_0^\alpha$  norm, then we are naturally lead to decompose as in eqn. (15), that is

$$\|h\|_{\tilde{H}^\alpha}^2 = |\text{avg}(h)|_{\mathbb{R}^n}^2 + \lambda \|\tilde{h}\|_{H_0^\alpha}^2 \quad (19)$$

This means that the two spaces  $\mathbb{R}^n$  and  $D_c M$  are orthogonal w.r.t.  $\tilde{H}^\alpha$ .

In the above,  $\mathbb{R}^n$  is akin to be the space of translations and  $D_c M$  the space of non-translating deformations. That labeling is not rigorous, though! since the subspace of  $T_c M$  that does not move the center of mass  $\text{avg}(c)$  is not  $D_c M$ , but rather

$$\left\{ h : \int_{S^1} h + (c - \text{avg}(c)) \langle D_s h \cdot T \rangle \, ds = 0 \right\}$$

according to eqn. (3).

The decomposition (19) is at the base of a two step alternating algorithm for minimization of  $\tilde{H}^1$  Sobolev Active Contours (see §3.4 in [SYM05a]), where the tracking of contours is done by, alternatively, minimizing an energy on curves translations, and then on curves deformations in  $D_c M$  with the metric  $\|\tilde{h}\|_{H_0^1}$ . The resulting algorithm is independent of the choice of  $\lambda$ .

Note that  $\sqrt{\langle h, h \rangle_{H_0^\alpha}}$  is a norm on  $D_c M$  (by (21)), and it is a seminorm and not a norm on  $T_c M$ .

We define norms as

$$F_{H^j}(c, h) = \|h\|_{H^j} = \sqrt{\langle h, h \rangle_{H^j}} \quad , \quad F_{\tilde{H}^j}(c, h) = \|h\|_{\tilde{H}^j} = \sqrt{\langle h, h \rangle_{\tilde{H}^j}}$$

and consequently we define distances  $d_{H^j}$  and  $d_{\tilde{H}^j}$  as explained in 10.

### §3.i Stuff

We improve a result from [SYM05a] <sup>(7)</sup>: we show that the norms associated with the inner products  $H^j$  and  $\tilde{H}^j$  are equivalent. We first prove

**Lemma 16 (Poincaré inequalities)** *Pick  $h : [0, L] \rightarrow \mathbb{R}^n$ , weakly differentiable, with  $h(0) = h(L)$  (so  $h$  is periodically extensible) so that*

$$h(u) - h(0) = \int_0^u h'(s) \, ds = - \int_u^L h'(s) \, ds$$

then derive these equations

$$\begin{aligned} h(u) - h(0) &= \frac{1}{2} \left( \int_0^u h'(s) \, ds - \int_u^L h'(s) \, ds \right) \Rightarrow \\ \Rightarrow \text{avg}(h) - h(0) &= \frac{1}{2L} \int_0^L \left( \int_0^u h'(s) \, ds - \int_u^L h'(s) \, ds \right) du \Rightarrow \\ \Rightarrow |\text{avg}(h) - h(0)| &\leq \frac{1}{2L} \int_0^L \left( \int_0^u |h'(s)| \, ds + \int_u^L |h'(s)| \, ds \right) du = \\ &= \frac{1}{2L} \int_0^L \left( \int_0^L |h'(s)| \, ds \right) du = \frac{1}{2} \int_0^L |h'(s)| \, ds \end{aligned}$$

so that (by extending  $h$  and replacing 0 with an arbitrary point)

$$\sup_u |h(u) - \text{avg}(h)| \leq \frac{1}{2} \int_0^L |h'(s)| \, ds \quad (20)$$

the constant  $1/2$  is optimal and is approximated by a family of  $h$  such that  $h'(s) = a(\mathbb{1}_{[0, \varepsilon)}(s) - \mathbb{1}_{[L-\varepsilon, L)}(s))$  when  $\varepsilon \rightarrow 0$  (for a fixed  $a \in \mathbb{R}^n$ ). <sup>(8)</sup>

By using Hölder inequality we can then derive many useful Poincaré inequalities of the form  $\|h - \text{avg}(h)\|_p \leq c_{p,q,j} \|h'\|_q$ . By Fourier transform we can also prove for  $p = q = 2$  that

$$\int_0^L |h(s) - \text{avg}(h)|^2 \, ds \leq \frac{L^{2j}}{(2\pi)^{2j}} \int_0^L |h^{(j)}(s)|^2 \, ds \quad (21)$$

where the constant  $c_{2,2,j} = (L/2\pi)^{2j}$  is optimal and is achieved by  $h(s) = a \sin(2\pi s/L)$  (with  $a \in \mathbb{R}^n$ ). <sup>(9)</sup>

**Proposition 17** *Fix a smooth immersed curve  $c : S^1 \rightarrow \mathbb{R}^n$ , let  $L = \text{len}(c)$ . By Hölder's inequality, we have that  $|\text{avg}(h)|^2 \leq \frac{1}{L} \int_0^L |h(s)|^2 \, ds$  so that  $\|h\|_{\tilde{H}^j} \leq \|h\|_{H^j}$ . On the other hand,*

$$\frac{1}{L} \int_0^L |h(s) - \text{avg}(h)|^2 \, ds = \frac{1}{L} \int_0^L |h(s)|^2 \, ds - |\text{avg}(h)|^2 \quad (22)$$

<sup>(7)</sup>and we provide a better version that unfortunately was prepared too late for the printed version of [SYM05a]

<sup>(8)</sup> $\mathbb{1}_A(x)$  is the characteristic function, taking value 1 for  $x \in A$ , 0 for  $x \notin A$ .

<sup>(9)</sup>This  $h$  is not the only solution; for  $n = 2$  we also have  $h(s) = (\cos(2\pi s/L), \sin(2\pi s/L))$ .

so that (by the Poincaré inequality (21)),

$$\begin{aligned}
\|h\|_{H^j}^2 &= \int_0^L \frac{1}{L} |h(s)|^2 + \lambda L^{2j-1} |h^{(j)}(s)|^2 ds \\
&= \frac{1}{L} \int_0^L |h(s) - \text{avg}(h)|^2 ds + \int_0^L \lambda L^{2j-1} |h^{(j)}(s)|^2 ds + |\text{avg}(h)|^2 \\
&\leq |\text{avg}(h)|^2 + L^{2j-1} \left( \frac{1}{(2\pi)^{2j}} + \lambda \right) \int_0^L |h^{(j)}(s)|^2 ds \leq \frac{1 + (2\pi)^{2j} \lambda}{(2\pi)^{2j} \lambda} \|h\|_{\tilde{H}^j}^2
\end{aligned}$$

Consequently,

$$d_{\tilde{H}^j} \leq d_{H^j} \leq \sqrt{\frac{1 + (2\pi)^{2j} \lambda}{(2\pi)^{2j} \lambda}} d_{\tilde{H}^j}$$

More in general

**Proposition 18** For  $i = 0, \dots, j$ , choose  $\bar{a}_0 \geq 0$  and  $a_i \geq 0$  with  $a_0 + \bar{a}_0 > 0$  and  $a_j > 0$ . Define a  $H^j$ -type Riemannian norm<sup>(10)</sup>

$$\|h\|_{(a),j}^2 \stackrel{\text{def}}{=} \bar{a}_0 |\text{avg}(h)|^2 + \sum_{i=0}^j a_i L^{2i-1} \int_0^L |h^{(i)}(s)|^2 ds \quad (23)$$

then all such norms are equivalent.

Moreover, choose  $r$  with  $1 \leq r \leq j$ , and choose  $\bar{b}_0 \geq 0$ ,  $b_i \geq 0$  with  $\bar{b}_0 + b_0 > 0$ ,  $b_r > 0$ : then the norm  $\|h\|_{(a),j}$  is stronger than the norm  $\|h\|_{(b),r}$ .

*Proof.* The proof is just an application of (22) and of (21) (repeatedly); note also that for  $1 \leq i < j$  equation (21) becomes

$$\int_0^L |h^{(i)}(s)|^2 ds \leq \frac{L^{2j-2i}}{(2\pi)^{2j-2i}} \int_0^L |h^{(j)}(s)|^2 ds \quad (24)$$

since  $\text{avg}(h^{(i)}) = 0$ . □

So our definitions of  $\|\cdot\|_{H^j}$  and  $\|\cdot\|_{\tilde{H}^j}$  are in a sense the simpler choices of a Sobolev type norm that are scale invariant; in particular,

**Remark 19** the  $H^j$  type metric

$$\|h\|_M^2 \stackrel{\text{def}}{=} \int_0^L \sum_{i=0}^j |h^{(i)}(s)|^2 ds$$

studied in [MM06a] is equivalent to our choices,

$$b_1 \|\cdot\|_{\tilde{H}^j} \leq \|\cdot\|_M \leq b_2 \|\cdot\|_{\tilde{H}^j}$$

but the constants  $b_1, b_2$  depend on the length of the curve.

Following this proposition, we will prove some properties of the  $H^1$  metric, and we will know that they can be extended to  $\tilde{H}^1$  and to more general  $H^j$ -type metrics defined as in (23).

We prove this fundamental inequality (25):

<sup>(10)</sup>the scalar product can be easily inferred

**Proposition 20** Fix a smooth immersed curve  $c : S^1 \rightarrow \mathbb{R}^n$ , let  $L = \text{len}(c)$ .

We rewrite for convenience

$$\|h\|_{H^1}^2 \geq \lambda L^2 \langle h', h' \rangle_{H^0} = \lambda L \int_0^L |h'(s)|^2 ds = \lambda \int |\dot{c}(u)| du \int |h'(u)|^2 |\dot{c}(u)| du$$

where  $h' = D_s h$ ; then by Cauchy-Schwartz

$$\int |\dot{c}(u)| du \int |h'(u)|^2 |\dot{c}(u)| du \geq \left( \int |h'(u)| |\dot{c}(u)| du \right)^2$$

Suppose now that  $C(u, v)$  is a smooth homotopy of immersed curves  $C(\cdot, v)$ : then set  $h(u, v) = \partial_v C(u, v)$  so that  $D_s h = D_s \partial_v C = \partial_{uv} C / |\partial_u C|$ . Summarizing the above

$$\|\partial_v C(\cdot, v)\|_{H^1} \geq \sqrt{\lambda} \int |\partial_{uv} C(u, v)| du \quad (25)$$

As argued in 18, the above result extends to all  $H^j$ -type norms (23).

We related the  $H^1$ -type metric to the  $L^\infty$  type metrics

**Proposition 21** The  $\tilde{H}^1$  metric is stronger than the  $L^\infty$  metric defined in 12.

As a consequence, by theorem 17 and 13, the  $H^j$  and  $\tilde{H}^j$  distances are lower bounded by the Fréchet distance (with appropriate constants depending on  $\lambda$ ).

*Proof.* Indeed, by (20) there follows

$$\begin{aligned} \sup_\theta |\pi_{N(\theta)} h(\theta)| &\leq \sup_\theta |h(\theta)| \leq |\text{avg}(h)| + \frac{1}{2} \int |h'| ds \leq \\ &\leq |\text{avg}(h)| + \frac{\sqrt{L}}{2} \sqrt{\int |h'|^2 ds} \leq \sqrt{2} \sqrt{|\text{avg}(h)|^2 + \frac{L}{4} \int |h'|^2 ds} \end{aligned}$$

( $\pi_N$  was defined in eqn. (12)). For example, choosing  $\lambda = 1/4$ ,

$$F_\infty(c, h) \leq \sqrt{2} \|h\|_{\tilde{H}^1}$$

□

We also establish relationship between the length  $\text{len}(c)$  of a curve and the Sobolev metrics:

**Proposition 22** Suppose again that  $C(u, v)$  is a smooth homotopy of immersed curves, let  $L(v) \stackrel{\text{def}}{=} \text{len}(C(\cdot, v))$  be the length at time  $v$ ; then

$$\partial_v L = \int \langle \partial_{uv} C, \frac{\partial_u C}{|\partial_u C|} \rangle du \leq \int |\partial_{uv} C| du \leq \frac{1}{\sqrt{\lambda}} \|C_v(\cdot, v)\|_{H^1}$$

by (25).

We have many interesting consequences:

•

$$|L(1) - L(0)| \leq \frac{1}{\sqrt{\lambda}} \text{Len}(C) \quad (26)$$

where the length  $\text{Len}(C)$  of the homotopy/path  $C$  is computed using either  $H^1$  or  $\tilde{H}^1$  (or using any metric as in (23) above, but in this case the constant in (26) would change).

- Define the length functional  $c \mapsto \text{len}(c)$  on our space of curves; embed the space of curves with a  $H^1$  metric; then the length functional is Lipschitz.
- The “zero curves” are the constant curves (that have zero length); these are points in the space of curves where the space of curves is, in a sense, singular; by the above, the “zero curves” are a closed set in the  $H^1$  space of curves, and an immersed curve  $c$  is distant at least  $\text{len}(c)\sqrt{\lambda}$  from the “zero curves”.

But the most interesting consequence is that

**Theorem 23 (Completion of  $B_1$  w.r.t.  $H^1$ )** *let  $d_{H^1}$  be the distance induced by  $H^1$ ; the metric completion of the space of curves is contained in the space of all rectifiable curves.*

*Proof.* This statement is a bit fuzzy: indeed  $d_{H^1}$  is not a distance on  $M$ , whereas in  $B$  objects are not functions, but classes of functions. So it must be intended “up to reparametrization of curves”, as follows.<sup>(11)</sup>

Let  $(c_n)_{n \in \mathbb{N}}$  be a Cauchy sequence. Since  $d_{H^1}$  does not depend on parametrization, we assume that all  $c_n$  are parametrized by arc parameter, that is,  $|\partial_\theta c_n| = l_n$  constant in  $\theta$ . By proposition 21, all curves are contained in a bounded region; since  $\text{len}(c_n) = 2\pi l_n$  by proposition 22 above, the sequence  $l_n$  is bounded. So the (reparametrized) family  $(c_n)$  is equibounded and equilipschitz: by Ascoli-Arzelà theorem, up to a subsequence, we obtain that  $c_n$  converges uniformly to a Lipschitz curve  $c$ , and  $|\partial_\theta c| \leq \lim_n l_n$ .  $\square$

We also prove that

**Theorem 24** *Any rectifiable planar curve is approximable by smooth curves according to the distance induced by  $H^1$ .*

*Proof.* Let  $c$  be a rectifiable curve, and assume that it is non-constant.

As a first step, we assume that  $c$  is not *flat*, that is, the image of  $c$  is not contained in a line in the plane. We sketch how we can approximate  $c$  by smooth curves. The precise arguments are in section 2.1.4 in [YM04b]; see in particular the proofs of 2.12 and 2.15 in the appendix. Since the metric is independent of rescaling, we rescale  $c$ , and assume that  $|\partial_\theta c| = 1$ .

We identify  $S^1$  with  $[0, 2\pi)$ . Let in the following  $L^2 = L^2([0, 2\pi])$ . We define the *measurable angle function* to be a function  $\tau : [0, 2\pi) \rightarrow [0, 2\pi)$  such that  $\partial_\theta c(\theta) = (\cos \tau(\theta), \sin \tau(\theta))$ . We define

$$S = \{\tau \in L^2([0, 2\pi]) \mid \phi(\tau) = (0, 0)\}$$

where  $\phi : L^2 \rightarrow \mathbb{R}^2$  is defined by

$$\phi_1(\tau) = \int_0^{2\pi} \cos \tau(s) ds, \quad \phi_2(\tau) = \int_0^{2\pi} \sin \tau(s) ds$$

(this is similar to what was done in Srivastava et al. works on “*Shape Representation using Direction Functions*”, see [KSMJ03]).

As proved in 2.12 in [YM04b],  $S$  is a manifold near  $\tau$ . As shown in the proof of 2.15 in [YM04b], there exists a function  $\pi : V \rightarrow S$  defined in a neighbourhood

<sup>(11)</sup>The concept is clarified by introducing the concept of *horizontality* in  $M$ , that we must unfortunately skip for sake of brevity

$V \subset L^2$  of  $\tau$  such that, if  $f(s) \in L^2$  is smooth, then  $\pi(f)(s)$  is smooth. Let  $f_n$  be a smooth approximation of  $\tau$ , with  $f_n \rightarrow \tau$  in  $L^2$ ; then  $g_n \stackrel{\text{def}}{=} \pi(f_n) \rightarrow \tau$ . Let then

$$G_n(\theta, t) \stackrel{\text{def}}{=} \pi(t\tau + (1-t)g_n)(\theta)$$

be a the projection on  $S$  of the linear path connecting  $\tau$  to  $g_n$ . Since  $S$  is smooth in  $V$ , then the  $L^2$  distance  $\|\tau - g_n\|$  is equivalent to the geodesic induced distance; in particular,

$$\lim_n \mathbb{E}_S G_n = 0$$

where

$$\mathbb{E}_S G \stackrel{\text{def}}{=} \int_0^1 \|\partial_t G(\cdot, t)\|_{L^2}^2 dt$$

is the action of the path  $G$  in  $S \subset L^2$ .

The above  $G_n$  can be associated to an homotopy by defining

$$C_n(s, t) \stackrel{\text{def}}{=} c(0) + \int_0^s (\cos(G_n(\theta, t)), \sin(G_n(\theta, t))) d\theta$$

note that  $C_n(s, 0) = c(s)$  and  $C_n(s, 1)$  is a smooth closed curve.

We now compute the  $\tilde{H}^1$  action of  $C_n$ ,

$$\mathbb{E}_{\tilde{H}^1}(C_n) \stackrel{\text{def}}{=} \int_0^1 \|\partial_t C_n\|_{H^1}^2 dt = \int_0^1 \int_0^{2\pi} |\partial_t C_n|^2 + |D_s \partial_t C_n|^2 ds dt$$

Since any  $C_n(\cdot, t)$  is by arc parameter, then  $D_s \partial_t C_n = \partial_{st} C_n$  so

$$D_s \partial_t C_n = N(s) \partial_t G_n(s, t)$$

where

$$N(s) \stackrel{\text{def}}{=} (-\sin(G_n(s, t)), \cos(G_n(s, t)))$$

is the normal to the curve; so the second term in the action  $\mathbb{E}_{H^1}(C_n)$  is exactly equal to  $\mathbb{E}_S(G_n)$ , that is,

$$\mathbb{E}_{H^1}(C_n) = \int_0^1 \|\partial_t C_n\|_{H^1}^2 dt = \int_0^1 \int_0^{2\pi} |\partial_t C_n|^2 ds dt + E_S(G_n)$$

We can also prove that  $\int_0^1 \int_0^{2\pi} |\partial_t C_n|^2 ds dt \rightarrow 0$ , so  $\mathbb{E}_{H^1}(C_n) \rightarrow 0$ , and then

$$\lim_n \text{Len}_{H^1}(C_n) = 0$$

As a second step, to conclude, we assume that  $c$  is *flat*, that is, the image of  $c$  is contained in a line in the plane; then, up to translation and rotation,

$$c(\theta) = (c_1(\theta), 0)$$

since  $c$  is by arc parameter,  $\dot{c}_1 = \pm 1$ . Let then  $f : [0, 2\pi]$  be smooth and with support in  $[1, 3]$  and  $f(2) = 1$ ; let moreover

$$C(\theta, t) \stackrel{\text{def}}{=} (c_1(\theta), tf(\theta))$$

so

$$|\partial_\theta C| = \sqrt{1 + (f'(\theta))^2} \geq 1$$

and then we can easily prove that

$$\text{Len}_{H^1}(C) < \infty$$

moreover, any curve  $C(\cdot, t)$  for  $t > 0$  is not flat, so it can be approximated by smooth curves  $\square$

### §3.ii The completion of $M$ according to $H^2$ distance

Let  $d(c_0, c_1)$  be the geometric distance induced by  $H^2$  on  $M$  (as defined in 10). Let  $E(c) \stackrel{\text{def}}{=} \int |D_s^2 c|^2 ds$  be defined on non-constant smooth curves. We prove that

**Theorem 25**  *$E$  is locally Lipschitz in  $M$  w.r.t.  $d$ , and the local Lipschitz constant depends on the length of  $c$ .*

*As a corollary, all non-constant curves in the completion of  $C^\infty(S^1 \rightarrow \mathbb{R}^n)$  according to the metric  $H^2$  admit curvature as a measurable function, and the curvature satisfies  $E(c) < \infty$ .*

*Viceversa, any non-constant curve admitting curvature in a weak sense and satisfying  $E(c) < \infty$  is approximable by smooth curves.*

The rest of this section is devoted to proving the above three statements.

Fix a curve  $c_0$ ; let  $L_0 \stackrel{\text{def}}{=} \text{len } c_0$  be its length.

By eqn. (26) and prop. 18, we know that the “length function”  $c \mapsto \text{len}(c)$  is Lipschitz in  $M$  w.r.t the distance  $d$ , that is,

$$|\text{len } c_0 - \text{len } c_1| \leq a_1 d(c_0, c_1)$$

where  $a_1$  is a positive constant (dependent on  $\lambda$ ).

Choose any  $c_1$  with  $d(c_0, c_1) < L_0/(4a_1)$ .

Let  $C(\theta, t)$  be a time varying smooth homotopy connecting  $c_0$  to (a reparametrization of)  $c_1$ ; choose it so that  $\text{Len } C < 2d(c_0, c_1)$ ; then  $\text{Len } C < L_0/(2a_1)$ .

Let  $L(t) \stackrel{\text{def}}{=} \text{len } C(\cdot, t)$  be the length of the curve at time  $t$ . Since at all times  $t \in [0, 1]$ ,  $d(c_0, C(\cdot, t)) < L_0/(2a_1)$ , then  $|L(t) - L_0| < a_1 L_0/(2a_1) = L_0/2$ ; in particular,

$$L_0/2 < L(t) < L_0 3/2.$$

By using this last inequality, we are allowed to discard  $L(t)$  in most of the following estimates.

We call  $\|f\| \stackrel{\text{def}}{=} \sqrt{\int |f(s)|^2 ds}$  and

$$N(t) \stackrel{\text{def}}{=} \|D_s^2 \partial_t C(\cdot, t)\| = \sqrt{\int |D_s^2 \partial_t C|^2 ds}$$

for convenience; using this notation, we recall that

$$\|\partial_t C\|_{H^2} = \sqrt{\lambda L(t)^3 N(t)^2 + \frac{1}{L(t)} \|\partial_t C\|^2};$$

so  $\|\partial_t C\|_{H^2} \geq \sqrt{\lambda} L^{3/2} N(t)$ .

Up to reparametrization in the  $t$  parameter, we can suppose that the path  $t \mapsto C(\cdot, t)$  in  $M$  is by (approximate) arc parameter, that is  $\|\partial_t C\|_{H^2}$  is (almost) constant in  $t$ ; so we assume, with no loss of generality, that  $\|\partial_t C\|_{H^2} \leq 2d(c_0, c_1)$  for all  $t \in [0, 1]$ , and then  $N(t) \leq a_2 d(c_0, c_1)$  where  $a_2 = 2/\sqrt{(L_0/2)^3 \lambda}$ .

We want to prove that

$$E(c_1) - E(c_0) \leq a_5 d(c_0, c_1)$$

where the constant  $a_5$  will depend on  $L_0$  and  $\lambda$ .

By direct computation

$$\partial_t E(C(\cdot, t)) = \int |D_s^2 C|^2 \langle D_s \partial_t C, D_s C \rangle ds + 2 \int \langle D_s^2 C, \partial_t D_s^2 C \rangle ds$$

we deal with the two addenda in this way:

i). by Poincaré inequality (20) we deduce

$$\sup_{\theta} |D_s \partial_t C| \leq \frac{1}{2} \int |D_s^2 \partial_t C| ds \leq \sqrt{L(t)} \sqrt{\int |D_s^2 \partial_t C|^2 ds} = \sqrt{L(t)} N(t)$$

since  $\text{avg}(D_s \partial_t C) = 0$ .

So we estimate the first term as

$$\int |D_s^2 C|^2 \langle D_s \partial_t C, D_s C \rangle ds \leq E(C) \sqrt{L(t)} N(t).$$

ii). The commutator of  $D_s$  and  $\partial_t$  is  $\langle D_s \partial_t c, D_s c \rangle D_s$ : indeed

$$\begin{aligned} \partial_t D_s &= \frac{1}{|\partial_{\theta} c|} \partial_{\theta} \partial_t + (\partial_t \frac{1}{|\partial_{\theta} c|}) \partial_{\theta} = D_s \partial_t - \frac{\langle \partial_t \partial_{\theta} c, \partial_{\theta} c \rangle}{|\partial_{\theta} c|^3} \partial_{\theta} = \\ &= D_s \partial_t - \langle D_s \partial_t c, D_s c \rangle D_s \end{aligned}$$

so

$$\begin{aligned} \partial_t D_s^2 C &= D_s \partial_t D_s C - \langle D_s \partial_t C, D_s C \rangle D_s^2 C = \\ &= D_s^2 \partial_t C - D_s (\langle D_s \partial_t C, D_s C \rangle D_s C) - \langle D_s \partial_t C, D_s C \rangle D_s^2 C = \\ &= D_s^2 \partial_t C - (\langle D_s^2 \partial_t C, D_s C \rangle D_s C) - (\langle D_s \partial_t C, D_s^2 C \rangle D_s C) - \\ &\quad - 2(\langle D_s \partial_t C, D_s C \rangle D_s^2 C) \end{aligned}$$

so (since  $|D_s C| = 1$ )

$$\|\partial_t D_s^2 C\| \leq 2\|D_s^2 \partial_t C\| + 3\|D_s^2 C\| \sup |D_s \partial_t C|$$

that yields an estimate of the second term

$$\int \langle D_s^2 C, \partial_t D_s^2 C \rangle ds \leq \sqrt{E(C)} \left( 2N(t) + 2\sqrt{E(C)} \sqrt{L(t)} N(t) \right)$$

by using Cauchy-Schwartz.

Summing up

$$|\partial_t E(C(\cdot, t))| \leq 2\sqrt{E(C)} N(t) + 3E(C) \sqrt{L(t)} N(t)$$

or, since  $\sqrt{x} \leq 1 + x$ ,

$$|\partial_t E(C(\cdot, t))| \leq 2N(t) + 2E(C)N(t) + 3E(C)\sqrt{L(t)}N(t)$$

We recall that  $N(t) \leq a_2 d(c_0, c_1)$ ,  $L(t) \leq L_0 3/2$ , so we rewrite the above as

$$|\partial_t E(C(\cdot, t))| \leq 2a_2 d(c_0, c_1) + 2E(C)a_2 d(c_0, c_1) + 3E(C)a_4 a_2 d(c_0, c_1)$$

with  $a_4 = \sqrt{L_0 3/2}$ . Apply Gronwall's Lemma to obtain

$$E(c_1) \leq \left( E(c_0) + 2a_2 d(c_0, c_1) \right) \exp \left( (2 + 3a_4) a_2 d(c_0, c_1) \right) .$$

Let

$$g(y) \stackrel{\text{def}}{=} \left( E(c_0) + 2a_2 y \right) \exp \left( (2 + 3a_4) a_2 y \right)$$

then  $E(c_1) \leq g(d(c_0, c_1))$ ; since  $g$  is convex, and  $g(0) = E(c_0)$ , then there exists a  $a_5 > 0$  such that  $g(y) \leq E(c_0) + a_5 y$  when  $0 \leq y \leq L_0/(4a_1)$ ; since we assumed that  $d(c_0, c_1) < L_0/(4a_1)$ , then

$$E(c_1) \leq E(c_0) + a_5 d(c_0, c_1) .$$

Note that  $a_5$  is ultimately dependent on  $L_0$  and  $\lambda$ .

This ends the proof of the first statement of 25.

To prove the second statement, let  $(c_n)_{n \in \mathbb{N}}$  be a Cauchy sequence. Since  $d_{H^1} \leq ad_{H^2}$ , then as in the proof of 23, we assume that, up to reparametrization and a choice of subsequence,  $c_n$  converges uniformly to a Lipschitz curve  $c$ .

Let  $L_0 = \text{len } c$ . We have assumed in the statement that  $c$  is non-constant; then  $L_0 > 0$ .

Again, the “length function”  $c \mapsto \text{len}(c)$  is Lipschitz, so we know that the sequence  $\text{len}(c_n)$  is Cauchy in  $\mathbb{R}$ , so it converges; moreover the “length function”  $c \mapsto \text{len}(c)$  is lower semicontinuous w.r.t. uniform convergence, so  $\lim_n \text{len}(c_n) \geq \text{len}(c) > 0$ . So we assume, up to a subsequence, that  $2L_0 \geq \text{len}(c_n) \geq L_0$

We proved above that, in a neighbourhood of  $c$  of size  $L_0/(8a_1)$ , the function  $E(c) \stackrel{\text{def}}{=} \int |D_s^2 c|^2 ds$  is Lipschitz; so we know that the sequence  $E(c_n)$  is bounded, and then (since curves are by arc parameter and  $\text{len}(c_n) \geq L_0$ ) the energy  $\int |\partial_\theta^2 c|^2 ds$  is bounded: then  $\partial_\theta c_n$  are uniformly Hölder continuous, so by Ascoli-Arzelà compactness theorems, up to a subsequence,  $\partial_\theta c_n(\theta)$  converges.

As a corollary we obtain that  $\lim_n \text{len}(c_n) = \text{len}(c)$ , that  $c$  is parametrized by arc parameter, and that  $D_s c_n(\theta)$  converges to  $D_s c(\theta)$ .

Since the functional  $\int |\partial_\theta^2 c_n|^2 ds$  is bounded in  $n$ , then by a theorem in [Bre86],  $c$  admits weak derivative  $\partial_\theta^2 c$  and  $\int |\partial_\theta^2 c|^2 ds < \infty$ , and equivalently,  $\int |D_s^2 c|^2 ds < \infty$ .

For the third statement, viceversa, let  $c$  be a rectifiable curve, and assume that it is non-constant, and  $E(c) < \infty$ . Since the metric is independent of rescaling, we rescale  $c$ , and assume that  $|\partial_u c(u)| = 1$ .

We express in Fouries series

$$c(u) = \sum_{n \in \mathbb{Z}} l_n \exp(inu) \quad (27)$$

(by equating  $S^1 = \mathbb{R}/2\pi$ ), then we decide that

$$C(u, t) \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z}} l_n \exp(inu - f(n)t) \quad (28)$$

with  $f(n) = f(-n) \geq 0$  and  $\lim f(n)/\log(n) = \infty$ ; (for example,  $f(n) = |n|$  or  $f(n) = (\log(|n| + 2))^2$ ): then  $C(\cdot, t)$  is smooth for any  $t > 0$

We want to prove that, for  $t$  small,  $C(\cdot, t)$  is near  $c$  in the  $H^2$  metric; to this end, let  $\tilde{C}$  be the linear interpolator

$$\tilde{C}(u, t, \tau) \stackrel{\text{def}}{=} (1 - \tau)c(u) + \tau C(u, t) = \sum_{n \in \mathbb{Z}} l_n e^{inu} (1 - \tau + \tau e^{-f(n)t}) \quad (29)$$

we will prove that

$$\int_0^1 \left( \int_{S^1} |\partial_\tau \tilde{C}|^2 ds + \lambda L^4 \int_{S^1} |D_s^2 \partial_\tau \tilde{C}|^2 ds \right) d\tau < \delta(t) \quad (30)$$

where  $\lim_{t \rightarrow 0} \delta(t) = 0$ , and  $L$  is the length of  $\tilde{C}(\cdot, t, \tau)$ .

We need some preliminary results:

- we prove that

$$\int_{S^1} |\partial_{uu} c - \partial_{uu} \tilde{C}|^2 du < \delta_1(t) \quad (31)$$

where  $\lim_{t \rightarrow 0} \delta_1(t) = 0$ , uniformly in  $\tau \in [0, 1]$ ; we write

$$\int_{S^1} |\partial_{uu} c - \partial_{uu} \tilde{C}|^2 du = 2\pi\tau^2 \sum_{n \in \mathbb{Z}} |l_n|^2 |n|^4 (1 - e^{-f(n)t})^2$$

and since

$$2\pi \sum_{n \in \mathbb{Z}} |l_n|^2 |n|^4 = E(c) = \int_{S^1} |\partial_{uu} c|^2 ds < \infty$$

and  $\lim_{t \rightarrow 0} (1 - e^{-f(n)t})^2 = 0$ , we can apply Lebesgue dominated convergence theorem.

- We prove that

$$|\partial_u c - \partial_u \tilde{C}| < \delta_2(t) \quad (32)$$

where  $\lim_{t \rightarrow 0} \delta_2(t) = 0$ , uniformly in  $u$  and  $\tau \in [0, 1]$ ; indeed

$$\begin{aligned} |\partial_u c - \partial_u \tilde{C}| &\leq \tau \sum_{n \in \mathbb{Z}} |l_n| |n| (1 - e^{-f(n)t}) \leq \\ &\leq \sqrt{\sum_{n \in \mathbb{Z}} |l_n|^2 |n|^4} \sqrt{\sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{n^2} (1 - e^{-f(n)t})^2} \end{aligned}$$

again we apply Lebesgue dominated convergence theorem.

- By the above we also obtain that for  $t$  small,

$$3/2 \geq |\partial_u \tilde{C}| \geq 1/2 \text{ uniformly in } \tau, u \quad (33)$$

- We can similarly prove that

$$|c - \tilde{C}| < \delta_3(t) \quad (34)$$

By direct computation

$$D_s^2 \partial_\tau \tilde{C} = \frac{\partial_{uu\tau} \tilde{C}}{|\partial_u \tilde{C}|^2} + \frac{\langle \partial_{uu} \tilde{C}, \partial_u \tilde{C} \rangle \partial_{u\tau} \tilde{C}}{|\partial_u \tilde{C}|^4}$$

but then, for  $t$  small, by (33),

$$|D_s^2 \partial_\tau \tilde{C}| \leq 4|\partial_{uu\tau} \tilde{C}| + 24|\partial_{uu} \tilde{C}| |\partial_{u\tau} \tilde{C}|$$

We use the fact that

$$\partial_{uu\tau}\tilde{C} = \partial_{uu}C - \partial_{uu}c, \quad \partial_{u\tau}\tilde{C} = \partial_uC - \partial_uc, \quad \partial_\tau\tilde{C} = C - c,$$

so by eqn. (31) and eqn. (32)

$$\int \int |D_s^2 \partial_\tau \tilde{C}|^2 ds d\tau \leq a_1(\delta_1(t) + E(c)\delta_2(t))$$

and by eqn. (34)  $\int \int |\partial_\tau \tilde{C}|^2 ds d\tau \leq 8\delta_3(t)$ . Eventually we combine all above to bound eqn. (30) by setting  $\delta(t) = a_2\delta_3(t) + \lambda a_2(\delta_1(t) + E(c)\delta_2(t))$ .

This concludes the proof.

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## References

- [Bre86] H. Brezis, *Analisi funzionale*, Liguori Editore, Napoli, 1986, (italian translation of *Analyse fonctionnelle*, Masson, 1983, Paris).
- [Ham82] Richard S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 1, 65–222. MR MR656198 (83j:58014)
- [KSMJ03] Eric Klassen, Anuj Srivastava, Washington Mio, and Shantanu Joshi, *Analysis of planar shapes using geodesic paths on shape spaces*, 2003.
- [MM05] Peter W. Michor and David Mumford, *Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms*, Documenta Math. (2005), no. 10, 217–245.
- [MM06a] ———, *An overview of the riemannian metrics on spaces of curves using the hamiltonian approach*, arXiv:math.DG/0605009, 2006.
- [MM06b] ———, *Riemannian geometris of space of plane curves*, J. Eur. Math. Soc. (JEMS) **8** (2006), 1–48.
- [MS04] Washington Mio and Anuj Srivastava, *Elastic-string models for representation and analysis of planar shapes*, Conference on Computer Vision and Pattern Recognition (CVPR), June 2004.

- [Mum] D. Mumford, *Slides of the gibbs lectures*, <http://www.dam.brown.edu/people/mumford/Papers/Gibbs.pdf>.
- [Sha05] J. Shah,  *$H^0$  type Riemannian metrics on the space of planar curves*, arXiv:math.DG/0510192, 2005.
- [SYM05a] Ganesh Sundaramoorthi, Anthony Yezzi, and Andrea Mennucci, *Sobolev active contours*, VLISM 2005, 2005, <http://vlsm05.enpc.fr/programme.htm>.
- [SYM05b] ———, *Sobolev active contours*, Tech. report, GaTech, 2005, <http://users.ece.gatech.edu/~ganeshs/sobolev/sobolev.html>, <http://users.ece.gatech.edu/~ganeshs/sobolev/pubs/techrep.pdf>.
- [SYM06] ———, *Tracking with sobolev active contours*, CVPR, IEEE Computer Society, 2006.
- [YM04a] A. Yezzi and A. Mennucci, *Conformal Riemannian metrics in space of curves*, 2004, EUSIPCO04, MIA <http://www.ceremade.dauphine.fr/~cohen/mia2004/>.
- [YM04b] ———, *Metrics in the space of curves*, arXiv (2004), arXiv:math.DG/0412454.
- [YM05] ———, *Conformal metrics and true “gradient flows” for curves*, ICCV 2005, 2005, <http://research.microsoft.com/iccv2005/>.
- [You98] Laurent Younes, *Computable elastic distances between shapes*, SIAM Journal of Applied Mathematics **58** (1998), 565–586.